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Probabilistic Formulation of Classical Mechanics

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Abstract

Starting axiomatically with a system of finite degrees of freedom whose logic $\mathscr{L}_c$ is an atomic Boolean σ-algebra, we prove the existence of phase space Ω_c, as a separable metric space, and a natural (weak) topology on the set of states $\mathscr{S}$ (all the probability measures on $\mathscr{L}_c$) such that Ω_c, the subspace of pure states $\mathscr{P}$, the set of atoms of $\mathscr{L}_c$ and the space $\mathscr{P}(\Omega_c)$ of all the atomic measures on Ω_c, are all homeomorphic. The only physically accessible states are the points of Ω_c. This probabilistic formulation is shown to be reducible to a purely deterministic theory.

I

This note treats the probabilistic theory of a system whose logic is a Boolean σ-algebra and shows its reduction to a completely deterministic one. The set of states $\mathscr{S}$ consists of all the probability measures on the logic, and the set of observables $\mathscr{O}$ consists of all the σ-homomorphisms on the real Borel σ-field into the logic. This is a special case of generalised quantum theory as defined by Kronfli (1970) which includes both quantum logics and σ-algebras as special cases. On the other hand, conventional quantum theory yields classical mechanics only as an approximation. The results obtained are not very surprising, although they are more detailed technically. The main point, however, is the conclusion that *a theory is deterministic if and only if its logic is a Boolean σ-algebra.* It is hoped that this will lend support to the probabilistic point of view, adopted in the lattice-theoretic formulation of the generalised theory, as a fruitful approach to the mathematical analysis of fundamental physics.

From this axiomatic formulation follows the existence of phase space as a separable metric space which is topologically and set theoretically equivalent to the set of all the pure states $\mathscr{P}$ of $\mathscr{S}$. In any state in $\mathscr{P}$ each observable is sharply defined with zero variance such that the states $\mathscr{S}\backslash\mathscr{P}$ become inaccessible physically.

II

The definition and existence of phase space follow from

Theorem 1

For a system with finite degrees of freedom whose logic $\mathscr{L}_c$ is a Boolean σ-algebra, there exist a separable metric space Ω and a σ-homomorphism φ on the Borel σ-field $\mathscr{B}(\Omega)$ of Ω onto $\mathscr{L}_c$.

Proof: The finiteness of the degrees of freedom implies the existence of a finite set of observables which is complete. Let $\mathscr{L}_1$ be the Boolean sub-σ-algebra of $\mathscr{L}_c$ generated by the ranges of these observables. Then $\mathscr{L}_1$ is countably generated, since the range of each observable is countably generated. Completeness means that $\mathscr{L}_1$ is maximal, and hence equals $\mathscr{L}_c$. Thus $\mathscr{L}_c$ is countably generated.

By Loomis theorem (Loomis, 1947) there exists a set X, a σ-algebra $\mathscr{A}_1$ of subsets of X and a σ-homomorphism h_1 of $\mathscr{A}_1$ onto $\mathscr{L}_c$. Let $(a_n) \subset \mathscr{L}_c$ generate $\mathscr{L}_c$. Since h_1 is onto, there exists a countable set $(A_n) \subset \mathscr{A}_1$ such that $a_n = h_1(A_n)$ $(n \in N)$. Let $\mathscr{A}_2$ be the sub-σ-algebra of $\mathscr{A}_1$ generated by (A_n). Then $h_1(\mathscr{A}_2)$ is a sub-σ-algebra of $\mathscr{L}_c$ containing its generators (a_n), and hence equals $\mathscr{L}_c$. Since $\mathscr{A}_2$ is countably generated, there exists a separable metric space Ω and a σ-isomorphism h_2 on $\mathscr{B}(\Omega)$ onto $\mathscr{A}_2$ (see Parthasarathy, 1967, p. 133, theorem 2.2). The proof is completed by putting $\varphi = h_1 \circ h_2$. ■

From now on $\mathscr{L}_c$ denotes a countably generated Boolean σ-algebra, for instance when it is a Boolean σ-algebra and the system has finite degrees of freedom. The set of all states will be denoted by $\mathscr{S}$ and the pure ones by $\mathscr{P}$. The space Ω is as in Theorem 1. Let $\Omega_c = \{x \in \Omega : \varphi(\{x\}) \neq \varnothing\}$. The separable metric space Ω_c will be called the *phase space* associated with $\mathscr{L}_c$. [The author is not aware if there exists a φ, as in Theorem 1, such that $\Omega_c \in \mathscr{B}(\Omega)$. In this case $\mathscr{B}(\Omega_c) = \{A \in \mathscr{B}(\Omega) : A \subset \Omega_c\}$. This would then make $\mathscr{B}(\Omega_c)$ and $\mathscr{L}_c$ σ-isomorphic. No such assumption is made here.]

Recall that a Boolean algebra is *atomic* if every non-atomic element $(\neq \varnothing)$ dominates at least one atom. It is essential that $\Omega_c \neq \varnothing$. The next result shows that atomicity of $\mathscr{L}_c$ is a sufficient condition. In a forthcoming paper we shall show that it is also necessary in order to make a Boolean system deterministic.

Theorem 2

Let $\mathscr{L}_c$ be atomic. Then $\Omega_c \neq \varnothing$. Furthermore, the mapping $\gamma : x \to \varphi(\{x\})$ is a bijection on Ω_c onto the set $\mathscr{A}$ of the atoms of $\mathscr{L}_c$.

Proof: Let $a_0 \in \mathscr{A}$ and (a_n) generate $\mathscr{L}_c$. For each n, either $a_0 < a_n$ or $a_0 < a_n'$. If necessary replace a_n by a_n' so that $a_0 < a_n$ for all n. Clearly (a_n)

still generates $\mathscr{L}_c$. Since φ is onto let $A_n \in \mathscr{B}(\Omega)$ such that $\varphi(A_n) = a_n$. Put $B = \bigcap_n A_n$. Then $\varphi(B) = \bigwedge_n a_n > a_0 \neq \varnothing$, implying $B \neq \varnothing$. Note that (A_n) generates $\mathscr{B}(\Omega)$. Now $\mathscr{R} = \{E \in \mathscr{B}(\Omega) : B \subset E \text{ or } B \cap E = \varnothing\}$ is a sub-σ-algebra of $\mathscr{B}(\Omega)$ containing its generators and hence equals $\mathscr{B}(\Omega)$. But since the latter contains all singletons, $\mathscr{R} = \mathscr{B}(\Omega)$ is possible only if $B = \{x\}$ for some $x \in \Omega$. But $\varphi(\{x\}) \neq \varnothing$, thus $x \in \Omega_c$ and $\Omega_c \neq \varnothing$.

Now let $x, y \in \Omega_c$, $x \neq y$ and $\gamma(x) = \gamma(y) = a$ say. Then $\{x\} \subset \{y\}'$ implying $a < a'$, i.e. $a = \varnothing$, which is a contradiction. Hence $x = y$ and γ is one-one on Ω_c.

Let $x \in \Omega_c$, $a_0 = \gamma(x)$, $a \in \mathscr{L}_c$ and $a < a_0$. Then there exists $A \in \mathscr{B}(\Omega)$ such that $\varphi(A) = a$. Now either $x \in A$ or $x \in A'$. Thus either $a_0 < a$ implying $a = a_0$; or $a_0 < a'$ implying $a < a'$ i.e. $a = \varnothing$. Hence a_0 is an atom. Thus γ is an injection on Ω_c into $\mathscr{A}$.

Let $a_0 \in \mathscr{A}$. As before we can choose a generating sequence (a_n) in $\mathscr{L}_c$ such that $a_0 < a_n$ for all n. Let A, $A_n \in \mathscr{B}(\Omega)$ such that $\varphi(A) = a_0$ and $\varphi(A_n) = a_n$. Let $B_n = A \cup A_n$. Then $\varphi(B_n) = a_n$ and, therefore, (B_n) generates $\mathscr{B}(\Omega)$. Put $B = \bigcap_n B_n$. Then $\varphi(B) \neq \varnothing$ and as in the first paragraph of the proof $B = \{x\}$ for some $x \in \Omega_c$. Clearly $x \in A$ and hence $A \cap \Omega_c \neq \varnothing$. Thus $\varnothing \neq \gamma(x) < a_0$ implying $\gamma(x) = a_0$. Take $y \in A \cap \Omega_c$ and $x \neq y$. Then $\varnothing \neq \gamma(y) < a_0$ implying $\gamma(x) = \gamma(y) = a_0$. But γ is one-one making $x = y$, a contradiction. Thus A is a singleton and γ is onto. ■

From now on $\mathscr{L}_c$ will denote a countably generated atomic Boolean σ-algebra.

Let $\mathscr{A}$ be the set of all atoms of $\mathscr{L}_c$. For each $a \in \mathscr{A}$, define q_a by

$$q_a(b) = \begin{cases} 1 & a < b \\ 0 & a < b' \end{cases} \qquad (b \in \mathscr{L}_c)$$

Clearly, $q_a \in \mathscr{S}$, since $\mathscr{L}_c$ is atomic. Let $\mathscr{M}(\Omega)$ be the set of all probability measures on $(\Omega, \mathscr{B}(\Omega))$. For each $x \in \Omega$ define δ_x to be the atomic measure $\in \mathscr{M}(\Omega)$ concentrated at x. Put $\mathscr{P}(\Omega_c) = \{\delta_x : x \in \Omega_c\}$. The next result shows the simple structure of the set of pure states $\mathscr{P}$.

Theorem 3

The set of pure states on $\mathscr{L}_c$ *is precisely* $\mathscr{P} = \{q_a : a \in \mathscr{A}\}$. *Furthermore,* φ *induces a bijection* $\hat{\varphi}$ *of* $\mathscr{P}$ *onto* $\mathscr{P}(\Omega_c)$.

Proof: The proof of the first part is very much the same as that by Varadarajan (1968, Theorem 6.6). For the second part define $\hat{\varphi} : p \to p \circ \varphi$ $(p \in \mathscr{S})$. Clearly, $\hat{\varphi}$ is a convex homomorphism on $\mathscr{S}$ into $\mathscr{M}(\Omega)$ mapping the extreme points $\mathscr{P}$ of $\mathscr{S}$ into the extreme points $\mathscr{P}(\Omega)$ of $\mathscr{M}(\Omega)$. To prove that $\hat{\varphi}$ is a bijection of $\mathscr{P}$ onto $\mathscr{P}(\Omega_c)$, let $x \in \Omega_c$, $\delta_x \in \mathscr{P}(\Omega_c)$ as defined above, and $a_x = \gamma(x)$.

Define $p_x = q_{a_x}$. Clearly

$$p_x(\varphi(A)) = \begin{cases} 1 & x \in A \\ 0 & x \in A' \end{cases} \qquad (A \in \mathscr{B}(\Omega))$$

Thus $\delta_x = p_x \circ \varphi = \hat{\varphi}(p_x)$, implying that $\hat{\varphi}$ maps $\mathscr{P}$ onto $\mathscr{P}(\Omega_c)$. Finally, let $q_a, q_b \in \mathscr{P}$ such that $\hat{\varphi}(q_a) = \hat{\varphi}(q_b) = \delta_x$, say. This implies that $\gamma^{-1}(a) = \gamma^{-1}(b) = x$. But γ is a bijection of Ω_c onto $\mathscr{A}$, and hence $a = b$ or, equivalently, $q_a = q_b$, proving that $\hat{\varphi}$ is one-to-one on $\mathscr{P}$. ∎

So far no use was made of the topological properties of Ω. All the results obtained would work for Ω as an abstract set and $\mathscr{B}(\Omega)$ a countably generated σ-field of subsets of Ω containing all singletons.

Let $\mathscr{M}(\Omega)$ be equipped with its weak topology. We define the *weak topology* on $\mathscr{S}$ to be the weakest such that $\hat{\varphi}$ is continuous on $\mathscr{S}$. Since Ω is a separable metric space, then $\mathscr{M}(\Omega)$ is metrisable as a separable metric space and the spaces Ω_c and $\mathscr{P}(\Omega_c)$ are homeomorphic (see Parthasarathy, 1967, pp. 42–43). We have now both topological and set theoretical equivalence of all three spaces Ω_c, $\mathscr{P}(\Omega_c)$ and $\mathscr{P}$. This topological equivalence is important when considering continuous groups of convex automorphisms of $\mathscr{S}$ and their induced representations for motions in Ω_c, in particular the (one-parameter) dynamical group.

III

In this section is shown that the points of Ω_c are the only physically accessible states of the system such that at each point every observable is sharply defined with zero variance. This depends on an important theorem of Varadarajan. Let $\mathbf{B}(\Omega, R)$ be the set of equivalence classes $[f]$ of all real-valued Borel functions f on Ω, where $f_1, f_2 \in [f]$ if and only if $\{x \in \Omega : f_1(x) \neq f_2(x)\} \in \mathrm{Ker}(\varphi)$. Let $\mathscr{O}$ be the set of observables on $\mathscr{L}_c$.

Theorem 4

There exists a mapping $f: u \to [f_u]$ *of* $\mathscr{O}$ *into* $\mathbf{B}(\Omega, R)$ *such that*

(i) $u(E) = \varphi(f_u^{-1}(E)) \qquad (E \in \mathscr{B}(R))$,

(ii) $f_u(x) = 0 \qquad$ *for all* $x \in \Omega_c'$.

Proof: See Varadarajan, 1968, Theorem 1.4. ∎

Corollary

Let $x \in \Omega_c$, $u \in \mathscr{O}$ *and* $[f_u]$ *be the corresponding element in* $\mathbf{B}(\Omega, R)$ *as in Theorem* 4. *Then the expectation value of* u *in the state* x *is* $f_u(x)$, *and its variance is zero.*

Proof: Let $p_x = \hat{\varphi}^{-1}(\delta_x)$ and $\mu_x: E \to p_x \circ \varphi(f_u^{-1}(E))$ $(E \in \mathscr{B}(R))$. Then, clearly, μ_x is the atomic measure on R concentrated at $f_u(x)$. The expectation value of u in the state p_x is

$$\int_{-\infty}^{+\infty} t p_x(u(dt)) = \int_{\infty}^{+\infty} t \mu_x(dt) = f_u(x)$$

The variance is

$$\int_{-\infty}^{+\infty} t^2 p_x(u(dt)) - f_u(x)^2 = 0 \blacksquare$$

Now any convex automorphism on $\mathscr{S}$ is a one-to-one mapping of $\mathscr{P}$ onto itself. This is the same for the dynamical group $\{U_t : t \in R\}$. Starting the system in a well defined state $p_x \in \mathscr{P}(x \in \Omega_c)$, its state will always remain in $\mathscr{P}$ for all time t.

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